Stability of Matter p.3

5. Stability of 1st kind

5. Stability of first kind

Notation: we will say that $f \in L^p + L^q$ if $f = f_1 + f_2$ where $f_1 \in L^p$, $f_2 \in L^q$.

Equipped with the Sobolev inequality we are ready to prove

Thm (stability of first kind, $d=3$)

Let $d=3$ and assume that

$$H = -\Delta + V \quad \text{with} \quad V_- \in L^{3/2}(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$$

Then $E_0 > -\infty$.

$$V_- = \max\{-V, 0\}$$

Proof.

Recall the Sobolev inequality in $d=3$:

$$\|\nabla f\|_{L^2}^2 \geq C_3 \|f\|_6^2$$

with C_3 a constant. The optimal constant is in fact $C_3 = \frac{3}{4}(4\pi^2)^{2/3}$.

Recall

$$\mathcal{E}(\varphi) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \varphi|^2 dx + \int_{\mathbb{R}^3} V(x)\, |\varphi(x)|^2 dx \, . \quad \text{Thus}$$

$$\mathcal{E}(\varphi) \geq \frac{C_3}{2} \|\varphi\|_6^2 + \int_{\mathbb{R}^3} V(x)\, |\varphi(x)|^2 dx \geq \frac{C_3}{2}\|\varphi\|_6^2 - \int_{\mathbb{R}} V_-(x) |\varphi(x)|^2 dx$$

We could now use Hölder's inequality

$$\boxed{\|fg\|_1 \leq \|f\|_p \|g\|_q \qquad \frac{1}{p}+\frac{1}{q}=1}$$

to obtain

$$\int_{\mathbb{R}^3} V_-(x)\,|\varphi(x)|^2 \leq \|V_-\|_{\frac{3}{2}}\,\|\varphi\|_6^2$$

Then

$$\mathcal{E}(\varphi) \geq \frac{C_3}{2}\|\varphi\|_6^2 - \|V_-\|_{\frac{3}{2}}\|\varphi\|_6^2 \geq 0$$

when $\|V_-\|_{3/2} \leq \frac{C_3}{2}$. (**)

Of course $\frac{1}{|x|}$ is not in $L^{3/2}$. This is why we need the L^∞ part. To this end we claim that there exists a constant $\lambda < 0$ such that

$$h(x) = \min\,(V(x) - \lambda, 0) \leq 0$$

satisfies $\|h\|_{3/2} \leq \frac{C_3}{4}$. (*) (proved below → exercise)

Let $V(x) = v_1 + v_2$ with $v_1 \in L^{3/2}$, $v_2 \in L^\infty$.

We have

$$\mathcal{E}(\varphi) = T_\varphi + \int_{\mathbb{R}^3} (v_1(x) - \lambda)\,|\varphi(x)|^2\,dx + \lambda \overbrace{\int_{\mathbb{R}^3} |\varphi(x)|^2 dx}^{=1}$$

$$+ \int v_2(x)\,|\varphi(x)|^2\,dx \geq \overbrace{\frac{1}{2}\left(T_\varphi + 2\langle\varphi, h\varphi\rangle\right)}^{\geq 0 \text{ by } (**)}$$

$$+ \frac{1}{2}T_\varphi + \lambda - \|v_2\|_\infty \geq \frac{1}{2}T_\varphi + \lambda - \|v_2\|_\infty > -\infty$$

$\blacksquare$

Exercise

$h(x) = \min(V(x) - \lambda, 0)$, $V(x) \le 0$, $V \in L^{\frac{3}{2}}$.

Then $\forall \varepsilon > 0$ $\exists$ exists a λ such that $\|h\|_{3/2} < \varepsilon$.

Solution:

$$\|h\|_{3/2}^{3/2} = \int_{\mathbb{R}^3} (-\overset{h \le 0}{\min}(V(x) - \lambda, 0))^{3/2} dx = \int_{V \le \lambda} (|V(x)| - |\lambda|)^{\frac{3}{2}}$$

$$\le \int_{|V| \ge |\lambda|} |V(x)|^{3/2} dx \longrightarrow 0 \quad \text{as } |\lambda| \to \infty$$

since $V \in L^{3/2}$

(indeed by monotone conv $\int |f| 1_{\{|f| \le n\}} \underset{n \to \infty}{\longrightarrow} \int |f|$) ∎

What about other dimensions?

The Sobolev inequality that we have proved works in $d > 2s$ where $s = 1$ in the non-relativistic case. Thus we can consider $d > 3$, too.

Exercise

Derive stability of first kind for non-relativistic matter in $d > 3$.

Solution

By the Sobolev inequality for $d \ge 3$

$$\|\nabla \varphi\|_2^2 \ge C_d \|\varphi\|_{\frac{2d}{d-2}}^2 .$$

In particular we have

$$\int_{\mathbb{R}^d} |V_-(x)|\,|\psi(x)|^2 dx \le \left(\int_{\mathbb{R}^d} |V_-|^{\frac{d}{2}}\right)^{2/d} \left(\int_{\mathbb{R}^d} |\psi|^{\frac{2d}{d-2}}\right)^{\frac{d-2}{d}}$$

$$\frac{2}{d} + \frac{1}{q} = 1 \qquad \frac{1}{q} = 1 - \frac{2}{d} = \frac{d-2}{d} \Rightarrow q = \frac{d}{d-2}$$

We can now repeat all steps from $d=3$ case to conclude that

$$\boxed{E_0 > -\infty \quad \text{if} \quad V \in L^{\frac{d}{2}}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d),\ d \ge 3}$$

Lower dimensions $d=1,2$

We now ask the same question for $d=1,2$.

Thm (Sobolev's inequality in $d=1,2$)

a) For any $f \in H^1(\mathbb{R})$ we have

$$\|f\|_\infty^2 \le \|f'\|_2 \, \|f\|_2$$

Furthermore

$$|f(x) - f(y)| \le \|f'\|_2 \, |x-y|^{1/2}.$$

(b) For any $f \in H^1(\mathbb{R}^2)$ we have

$$C_{2,p} \|f\|_p^2 \le \|\nabla f\|_2^2 + \|f\|_2^2 \quad \forall p \in [2,\infty)$$

Sketch of proof

(a) Assume $f \in C_0^\infty(\mathbb{R})$. Then

$$f(x)^2 = \int_{-\infty}^{x} f(y)f(y)'\,dy - \int_{x}^{\infty} f(y)f(y)'\,dy$$

Thus

$$|f(x)|^2 \le \int_{-\infty}^{x} |f||f'| + \int_{x}^{\infty} |f||f'| = \int_{-\infty}^{\infty} |f||f'|$$

$$\overset{CS}{\Rightarrow} \quad \|f\|_\infty^2 \le \|f'\|_2 \, \|f\|_2$$

One then uses a density argument since C_0^∞ is dense in H^1.

The other bound follows from the fact that

$$|f(x)-f(y)| = \left|\int_y^x f'(t)\,dt\right| \le \left(\int_y^x dt\right)^{1/2} \left(\int_y^x |f'(t)|^2\,dt\right)^{1/2} \le \sqrt{x-y}\,\|f\|_2$$

this follows since if $f \in H^1(\mathbb{R}) \Rightarrow f(x) = c + \int_a^x f'(x)\,dx$ a.e.

(obvious for smooth fcts).

b) Let $q \in (1,2)$. Then

$$\|\hat f\|_q^q = \int_{\mathbb{R}^2} |\hat f(k)|^q\,dk = \int_{\mathbb{R}^2} \frac{|\hat f(k)|^q (1+4\pi^2|k|^2)^{\frac{q}{2}}}{(1+4\pi^2|k|^2)^{q/2}}\,dk$$

$$\le \underbrace{\left(\int_{\mathbb{R}^2} |\hat f(k)|^2 (1+4\pi^2|k|^2)\,dk\right)^{q/2}}_{H^1 \text{ norm} \wedge 2} \left(\int_{\mathbb{R}^2} \frac{dk}{(1+4\pi^2|k|^2)^{\gamma}}\right)^{1-\frac{q}{2}}$$

where $\frac{1}{2/q} + \frac{1}{z} = 1$ $\qquad h = \frac{q}{2}\cdot z = \frac{q}{2}\frac{2}{2-q} = \frac{q}{2-q} > 1$

$$\frac{1}{z} = 1 - \frac{q}{2} \Rightarrow z = \frac{2}{2-q}$$

This implies $\displaystyle\int_{\mathbb{R}^2} \frac{dk}{(1+4\pi^2|k|^2)^h}\, dk < \infty$

$$\Rightarrow \quad \|\hat{f}\|_q^q \le C\|f\|_{H^1}^q \quad \Rightarrow \quad \|\hat{f}\|_q \le C\|f\|_{H^1}$$

Using the Hausdorff-Young inequality $\|f\|_p \le \|\hat{f}\|_q$ for $\frac{1}{p}+\frac{1}{q} = 1$, $q\in[1,2]$. Since $1<q<2 \Rightarrow p\in(2,\infty)$.
For $p=2$ it is obviously true. ∎

Exercise

Show the following theorem:

Thm (Stability of matter in lower dimensions)

Let $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ in $d=1$ and $V\in L^{1+\varepsilon}(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ in $d=2$. Then

$$E_0 > -\infty.$$

Proof / solution

We follow the argument for $d\ge 3$. This time we have

$\underline{d=1}$

$$\int_{\mathbb{R}} |V(x)| \, |\varphi(x)|^2 \, dx \le \|V\|_1 \, \|\varphi\|_\infty^2$$

and $\quad T\varphi \ge c \, \|\varphi(x)\|_\infty^4$

$$\Rightarrow \quad \mathcal{E}(\varphi) \ge c \, \|\varphi(x)\|_\infty^4 - \|V_-\|_1 \, \|\varphi\|_\infty^2 > -\infty$$

Note that here using the L^∞ is not necess but allows for more potentials.

$\underline{d=2.}$

Since $\quad \|\nabla f\|_2^2 \ge C_{d,p} \, \|f\|_p^2 - \|f\|_2^2$

$$\Rightarrow \quad \mathcal{E}(\varphi) \ge c \, \|\varphi\|_p^2 - c - \int_{\mathbb{R}^2} |V_-(x)| \, |\varphi(x)|^2 \, dx$$

if $\quad V \in L^\infty \Rightarrow$ take $p=2$ and done

if $\quad V \in L^{1+\varepsilon}$

$$\int_{\mathbb{R}^2} |V_-| \, |\varphi|^2 \le \left(\int_{\mathbb{R}^2} V^{1+\varepsilon} \right)^{\frac{1}{1+\varepsilon}} \left(\int |\varphi|^{2\tilde{p}} \right)^{\frac{1}{\tilde{p}}}$$

$$\frac{1}{1+\varepsilon} + \frac{1}{\tilde{p}} = 1 \Rightarrow \frac{1}{\tilde{p}} = 1 - \frac{1}{1+\varepsilon} = \frac{\varepsilon}{1+\varepsilon}$$

$$\Rightarrow \tilde{p} = \frac{1+\varepsilon}{\varepsilon} \xrightarrow[\varepsilon \to 0]{} \infty$$

So for $\forall \varepsilon > 0 \ \exists \ \tilde{p} < \infty$: $\quad 2\tilde{p} = q \quad \frac{1}{\tilde{p}} = \frac{2}{q}$

$$\mathcal{E}(\varphi) \ge c \, \|\varphi\|_{\tilde{p}}^2 - c - \tilde{C} \, \|\varphi\|_{\tilde{p}}^2.$$

We use the λ- trick L^∞ to get the bound since then we can make $\tilde{C} < c$. ∎